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Parastatistics, highest weight osp(N, ∞) modules, singleton statistics and confinement

M. FLATO

Physique Mathématique
Université de Bourgogne
BP 138, F21004 Dijon Cedex, France

C. FRONSDAL

Physics Department
University of California
Los Angeles, CA, 90024, USA

Abstract. *This paper reviews the development of parastatistics and interprets the para-Fock spaces as highest weight modules of infinite dimensional super Lie algebras. Several generalizations are proposed and the relation to composite models and confinement is pointed out. A concept of «kinematical confinement» is suggested, with possible application to the strong interactions.*

1. INTRODUCTION

The possibility of «abnormal» quantum statistics was raised long ago and occasionally become a hot subject. The idea of a quantum field theory based on something other than Bose-Einstein or Fermi-Dirac quantization became especially interesting at the time when physicists were debating the observability of quarks. Since that time, for what we think are excellent reasons, the idea of abnormal statistics (later replaced by «color») has been closely associated with confinement.

The basis of quantum statistics is the theory of canonical commutation relations. (In

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this paper, for reasons of economy, we limit ourselves entirely to generalizations of Bose-Einstein statistics; a parallel discussion of the Fermi-Dirac case would be repetitive). The idea of generalizations arises from the observation that the most important observables (free Hamiltonians, symmetry generators,. . . .) are bilinears in the fields (phase space variables). Seen in this (group theoretical) setting, it is immediately evident that «parastatistics», as introduced by Green and others, is susceptibile to immediate, natural and interesting generalizations. See especially what can be done with a multi-dimensional «vacuum» sector, below and in Section 4.

The current interest of the authors in reviewing these ideas stems from their work on singletons. The kinematics of singleton states, and more especially the structure of singleton gauge theory, shows that the singleton field is not locally observable, a property that is very similar to confinement. For singleton field theory to become physically relevant it is necessary to introduce a non local element into the theory; this can be done via the choice of statistics. In fact, non-observability of the basic field variables may be seen as an opportunity for the exploitation of generalized statistics. If micro causality is to be preserved, then quarks and gluons (dynamically confined) and singletons (kinematically confined) are the only particles for which non standard quantization rules may be contemplated. The quantization scheme that has been suggested for singleton field theory, to be described only briefly at the end of this review, differs from parastatistics in one important respect. The kinematical properties of two-singleton states are those of massless particles composed of singletons. The new quantization rules give rise to an enlarged Fock space (as in the case of parastatistics). The additional states, not the usual symmetrical two-singleton states, are identified as massless particles (a photon cannot decay into two singletons!). Our new quantization scheme (a deformation of Bose-Einstein quantization) insures that these massless particles obey standard statistics, something that is not guaranteed by parastatistics.

There is an obvious relationship between the fundamental space-time group (Poincaré, De Sitter, . . .) and the algebra of observables of the theory (sp , osp , . . .) : the former must act by automorphisms on the latter. Some of the implications of this are taken up in Section 6. One of the generalizations discussed in this paper deals with an infinite dimensional vacuum sector that is a carrier space for a U.I.R. of the De Sitter group. The «charge» that distinguishes between different vacua is in this case the De Sitter minimal energy E_0 . Such quantization schemes may be relevant to mass generation of the Higgs-Kibble type. Non-standard vacua of this kind may also find application to the cosmic residual background radiation.

As we have said already, abnormal statistics is accompanied by a violation of micro causality. This can be accepted in a physical theory only on the condition that the associated particles (quarks or singletons) be confined. This amounts to a limitation on the algebra of true physical observables, which should contain only micro-causal fields. Only particles connected to true observables can be directly observed. To emphasize this point, recall that the type of abnormal statistics (anticommutativity of different Bose fields) that is sometimes discussed within the context of Wightman axiomatic field theory can be removed by a *non-local* Araki-Klein transformation. Even such a mild deviation from standard statistics as this has profound consequences for the local (pointlike) behaviour of the fields.

Through string theories the concept of highest weight modules of infinite dimensional (super) Lie algebras has become of interest to physicists. The infinite dimensional orthosymplectic algebra osp($1,\infty$) is one of the simplest ones, and it is of direct interest in ordinary field theory. The unitarizable highest weight modules with a nondegenerate highest weight space (vacuum) are precisely the Fock spaces of parastatistics, and thus connected to QCD. The other unitarizable highest weight modules have degenerate vacua, but for all that they may well have physical applications, perhaps to symmetry breakdown.

Parastistics is by no means the only alternative to Bose-Einstein or Fermi-Dirac statistics. We review the subject in Sections 2-4 and propose some other possibilities in Section 5. Then we turn to a consideration of parastatistics in the framework of gauge theories. In Sections 7 and 8 we discuss compositeness and confinement, with an application to composite electrodynamics and a suggestion for the strong interactions.

2. STRUCTURE

One considers a field ϕ on space-time, with a discrete Fourier expansion

$$\sum_i \{\phi^j(x)a_j + \overline{\phi^j(x)}a^{*j}\}. \tag{2.1}$$

The functions ϕ^j are solutions of some wave equation, taking their values in a finite dimensional vector space. In a quantum field theory a_j and a^{*j} are operators acting in a Hilbert space. It is sometimes said that ϕ, and therefore also a_j and a^{*j}, are not directly observable. The most important observables, including the Hamiltonian, are bilinears (currents)

$$Q_j^k = \frac{1}{2}[a_j, a^{*k}]_+ \tag{2.2}$$

and perhaps also

$$P_{jk} = \frac{1}{2}[a_j, a_k]_+, \quad P^{*jk} = \frac{1}{2}[a^{*j}, a^{*k}]_+. \tag{2.3}$$

Bose-Einstein quantization is based on the postulate of canonical commutations relations

$$(2.4)\qquad \begin{aligned}&[a_j, a^{*k}]_- = \delta_j^k,\\ &[a_j, a_k]_- = [a^{*j}, a^{*k}]_- = 0.\end{aligned}$$

From this it follows that, for example,

$$(2.5)\qquad [Q_j^k, a^{*l}]_- = \delta_j^l a^{*k},$$

$$(2.6)\qquad [P_{jk}, a^{*l}]_- = \delta_j^l a_k + \delta_k^l a_j,$$

$$(2.7)\qquad [P_{jk}, a_l]_- = 0.$$

Now it can be argued that these relations are all that matter and that Eqs. (2.4) have a lower standing.

Eqs. (2.4) lead to a quantum field theory of particles obeying Bose-Einstein statistics. It was proposed by Gentile, [1] in 1940, that alternative forms of statistics, intermediate between Fermi-Dirac and Bose-Einstein, could be envisaged. He showed that a gas obeying intermediary statistics would show a characteristic thermodynamic behavior. This proposal was strongly criticized by Sommerfeld [2] and others [3] who argued that it was inconsistent with the basic structure of quantum mechanics. This is less surprising than the fact that the quantum mechanics of para-bosons remains poorly understood to this day.

In 1950, Wigner remarked [4] that, in the case of a system of one degree of freedom, specifically the harmonic oscillator, only Eq. (2.5) is needed to derive the correct equations of motion, and (2.4) can in fact be modified in an essential way. The structure investigated by Wigner is defined by

$$[Q, a]_- = -a, \quad [Q, a^*]_- = a^*, \quad [a, a^*]_+ = 2Q.$$

The super Lie algebra generated by a, a^*, Q and a^2, a^{*2}, with commutation relations for the last two dictated by the Jacobi identity, is $\mathrm{osp}(1,2)$, and Wigner was perhaps the first to study its representations.

These ideas – Gentile's statistics and Wigner's quantization postulate – came together for the first time in 1953, in the work of H.S. Green [5]. He took Eq. (2.5) as the basic postulate – necessary to derive the equations of motion– but added Eqs. (2.6) and (2.7) as well, perhaps because the operators P_{jk} would seem to be qualified observables. Anyway, the structure (2.5)-(2.7) has become the definition of parabose quantization.

It should be stressed that Eqs. (2.2) and (2.3) are interpreted as definitions of the $P's$ and $Q's$. It then follows from Eqs. (2.5)-(2.7) that the commutator algebra of these operators has the Lie algebra structure of the infinite sympletic Lie algebra denoted $\mathrm{sp}(\infty)$. This was first pointed out by Kamefuchi and Takahashi [6].

When this Lie algebra is extended by inclusion of the operators a_j, a^{*j} , with commutation relations (2.5)-(2.7) and the anti-commutation relations (2.2) and (2.3), then one recognizes the structure of the infinite orthosymplectic super Lie algebra $\mathrm{osp}(1,\infty)$. This was first pointed out by Omote et al. [7] in 1976. It should be noted that neither of these structures, $\mathrm{sp}(\infty)$ or $\mathrm{osp}(1,\infty)$, includes the canonical commutation relations (2.4).

The structure of parabose statistics is thus the Lie superalgebra $\mathrm{osp}(1,\infty)$. The special relations (2.4) are outside the structure and are defined in the enveloping algebra; they characterize a very special class of representations of $\mathrm{osp}(1,\infty)$ and this class includes the usual Fock space construction of Bose-Einstein field theory. Among the other hermitian representations of $\mathrm{osp}(1,\infty)$ the simplest ones are intimately related to parabose statistics, as will be seen next in Section 3. The remaining hermitian representations are constructed on degenerate vacua, in Section 4. After that we shall consider interesting alternatives to $\mathrm{osp}(1,\infty)$, for this algebra is not the only one on which one can base field quantization.

3. REPRESENTATIONS

A complete accounting of the linear representations of $\mathrm{osp}(1,\infty)$ is both difficult and unnecessary. Physics is often said to be interested exclusively in hermitian representations, but this is not true unless one would exclude gauge theories (including strings) from physics. What is true is that hermitian representations appear as subquotients of the representations that are realized on the field modes. The latter can sometimes be constructed from the former, but additional input is required, usually locality.

Fortunately, there is another important physical requirement that holds even in gauge theories. The energy should be positive or, more precisely, bounded below. Thus, we make the

POSITIVE ENERGY HYPOTHESIS. The spectrum of the energy operator H is bounded below.

If the spectrum of H is discrete, then there will be a subspace of Hilbert space on which H takes its lowest value; this subspace is usually taken to be one-dimensional, in which case one is led to the:

STRONG VACUUM HYPOTHESIS. There is a unique, one-dimensional subspace (with basis $|0\rangle$) on which the energy operator H reaches its lowest bound.

In the case of a relativistic field theory in Minkowski space the spectrum of H is never discrete. In this respect the situation is much more favorable in De Sitter space, where the energy spectrum of positive energy representations is always discrete. In Minkowski field theories without massless particles the lower bound of the spectrum is isolated. But if massless particles are present, as it is in all gauge theories, then the lower bound is at the end of a continuum (the soft photon spectrum), and in this case it is not at all obvious that the vacuum sector exists at all. For this reason we prefer to continue the discussion in the context of field theory on De Sitter space. *It should be noted that most of the cited papers deal with a discrete basis in the space of one particle states, something that is more than a little akward to justify in a Minkowski space field theory.*

A standard procedure allows us to construct all the irreducible representations of $\mathrm{osp}(1,n)$ with positive energy; *this includes all the hermitian representations* [8]. The method is well known as holomorphic induction and relies on the identification of a solvable subalgebra of $\mathrm{osp}(1,n)$. In the case of $\mathrm{osp}(1,\infty)$ the same construction can be carried out, but exhaustivity is not assured.

We suppose that there is an action of the space-time group $\mathrm{so}(3,2)$ on the linear space spanned by (a^{*j}), $j = 1,2,\dots$, and that this action is a positive energy representation. This justifies our referring to the $a^{*\prime}s$ as positive energy operators. The $P's(P^{*\prime}s)$ are negative (positive) energy operators, while the $Q's$ are of both kinds. The inducing subalgebra is

$$B = \mathrm{span}(a_j, P_{jk}, Q_j^k). \tag{3.1}$$

We choose an irreducible representation π_0 of B, on a space V_0, such that

$$\pi_0(a_j) = 0, \quad \pi_0(P_{jk}) = 0.$$

The restriction of π_0 to the subalgebra spanned by the Q_j^k is irreducible but otherwise arbitrary for the moment.

Let π be the representation of $\mathrm{osp}(1,\infty)$ that is induced from the representation π_0 of B;

$$\pi = \mathrm{IND} \overset{\mathrm{osp}(\infty)}{\underset{B}{\uparrow}} \pi_0 .$$

It is given by the natural action of $\mathrm{osp}(1,\infty)$ on the space

$$V = U \underset{B}{\otimes} V_0, \tag{3.2}$$

where U is the universal enveloping algebra of $\mathrm{osp}(1,\infty)$.

The space V_0 is the «vacuum» sector. At first we limit ourselves to the case when V_0 is one-dimensional (case of nondegenerate vacuum) and choose a basis $|0\rangle$ for V_0, the vacuum state. Writing e.g., a_j for $\pi(a_j)$, we then have

$$a_j|0\rangle = 0, \quad P_{jk}|0\rangle = 0,$$
$$Q_j^k|0\rangle = \frac{\lambda}{2}\delta_j^k|0\rangle, \quad \lambda \text{ real}. \tag{3.3}$$

The last equation (with λ integer) is the basic assumption, along with Green's commutation relations, for parabose statistics. This was first made explicit by McCarthy [9].

The vectors

$$|0\rangle, \quad a^{*j}|0\rangle, \quad a^{*j}a^{*k}|0\rangle, \quad \ldots$$

span V. We next introduce, in standard fashion, an invariant norm on V. Ignoring zero norm states we shall then recover ordinary Fock space when $\lambda = 1$ and para-Fock space when $\lambda = 2, 3, \ldots$.

Let x be a homogeneous polynomial in the $a^{*\prime}s$; then one easily shows (since $|0\rangle$ is a highest weight vector) that $x^*x|0\rangle$ is a real multiple $N(x)$ of $|0\rangle$. This gives a norm $N(x) = ||x||^2$ on V, and an inner product with respect to which the restriction of π to the real subsuperalgebra is hermitian. But this norm is not positive in general. We recall the analogus construction for osp$(1, 2n)$ [10]. For $\lambda > n-1$ the representation π is irreducible, and the restriction to osp$(1, 2n, \mathbb{R})$ is unitarizable. For $\lambda \leq n-1$ the representation is irreducible and not unitarizable, except for the values $\lambda = 0, 1, \ldots, n-1$. Thus, for infinite n only the non-negative, integral values of λ are of interest.

When λ belongs to this exceptional set, $\lambda = 0, 1, \ldots, n-1$ for osp$(1, 2n)$ and $\lambda = 0, 1, 2, \ldots$ for osp$(1, \infty)$, then the invariant norm is degenerate; V contains an invariant subspace V_g, the radical of the inner product. An irreducible, hermitian representation of the real subalgebra is induced on the quotient V/V_g. We shall demonstrate this for the lowest values of λ. The fact that parabose statistics requires λ to be a positive integer was postulated but not proved already by Green [5]. Greenberg and Messiah [11] seem to have been the first to connect integrality of λ to positivity of the norm.

Setting the norm $\langle 0|0\rangle$ to 1 we have

$$\langle 0|a_j a^{*k}|0\rangle = \langle 0|2Q_j^k|0\rangle = \lambda\delta_j^k,$$
$$\langle 0|a_j a_k a^{*l}a^{*m}|0\rangle = \lambda\{(2-\lambda)\delta_j^l\delta_k^m + \lambda\delta_j^m\delta_k^l\}, \tag{3.4}$$

which shows that:

a) When $\lambda = 0$, $a^{*j}|0\rangle$ and all other basis vectors except $|0\rangle$ have zero norm. The quotient V/V_g is one-dimensional and carries the trivial representation.

b) When $\lambda = 1$, the antisymmetric states $(a^{*l}a^{*m} - a^{*m}a^{*l})|0\rangle$ have zero norm. Less obvious at first sight is the fact that

$$[a^{*l}, a^{*m}]_- = 0 \tag{3.5}$$

on V/V_g; this is proved by induction. If

$$\Psi[j_1, \ldots, j_p] \equiv a^{*j_1} \cdots a^{*j_p}|0\rangle$$

is equal Mod V_g to its symmetric part (which, as we have just seen, is true for $p = 2$), then

$$a_j \Psi[j_1, \ldots, j_{p+1}]$$

is equal Mod V_g to

$$\sum_{k=1}^{p+1} \delta_j^{j_k} \Psi[j_1, \ldots, \hat{j}_k, \ldots, j_{p+1}],$$

which shows that $\Psi[j_1, \ldots, j_{p+1}]$ is equal Mod V_g to its symmetric part.

It is now easy to see that the first of Eqs. (2.4) also holds in V/V_g, when $\lambda = 1$. The space is thus ordinary Bose-Einstein Fock space. This representation of $osp(1,2n)$, with n finite or not, is called the oscillator representation; it is the most degenerate representation and, for finite n, the only one in which (2.4) holds.

For $\lambda = 2$, the vectors (3.4) have positive norm. On the other hand

$$a_j a^{*k} a^{*l} a^{*m}|0\rangle = (\delta_j^k a^{*l} a^{*m} + \delta_j^m a^{*l} a^{*k})|0>$$

is the symmetric in k, m, from which it may easily be deduced that the vectors

$$[a^{*k} a^{*l} a^{*m} - (k, m)]|0\rangle$$

have zero norm. Further calculations show that the quotient V/V_g is in this case the Fock space of the simplest form of para-Bose statistics. The following facts summarize, in the present idiom, the result of many people; the most complete statements are to be found in the book of Ohnuki and Kamefuchi, Ref. 11.

1) For any positive integer m, the action of the symmetric group $S(m)$ on the space V_m spanned by

$$a^{*j_1} \cdots a^{*j_m}|0\rangle$$

reduces to a direct sum of irreducible representations, identified with their Young tableaux. All tableaux with m boxes appear each one with multiplicity 1 [12].

2) When λ is a positive integer, then V/V_g coincides with the Fock space of para-Bose quantization of order λ [11].

3) When λ is a positive integer, then the intersection of V_m with the subspace V_g of zero norm contains precisely those Young tableaux that have more than λ rows. That is, the quotient $V_m/(V_m \cap V_g)$ of physical states contains every Young tableau with m boxes and no more than λ rows precisely once. See Ohnuki and Kamefuchi, [11] Bracken and Green [13].

The representations of the real subalgebra of $\mathrm{osp}(1,\infty)$ induced on V/V_g, in all cases with a nondegenerate vacuum and λ integer, are hermitian. A simple proof related to Green's ansatz is given in the Appendix.

The generalization of Bose-Einstein quantization proposed by Green [5] and subsequently known as parastatistics, is timid. The present view of it, in terms of highest weight $\mathrm{osp}(1,\infty)$ modules, carries us much further; to other types of highest weight modules (Section 4) and other quantization algebras (from Section 5 onward).

4. REPRESENTATIONS WITH DEGENERATE VACUA

Such representations may find applications in connection with broken symmetries, but other applications can also be envisaged. The «vacuum» of the representation space V is the subspace (that is canonically identified with) V_0. There is no reason, a priori, that this should be one-dimensional. In other words, the representation π_0 of the subalgebra B need not be one-dimensional. We do require that $\pi_0(a_j)$ and $\pi_0(P_{jk})$ vanish, but $\pi_0(Q_j^k)$ may be any hermitian representation. [The Q's span $u(n)$ inside $\mathrm{osp}(1,2n)$]. The simplest nontrivial example is to let V_0 be defined as the linear span of basis vectors $|j\rangle$, $j = 1,2,\ldots$, with the action

$$\pi_0(Q_j^k)|l\rangle = \delta_j^l|k\rangle + \frac{\lambda}{2}\delta_j^k|l\rangle. \tag{4.1}$$

Physically, this means that the «vacuum» contains precisely one particle; there is no state with zero particles. Note that the states $|j\rangle$ are not degenerate with respect to the energy. The true physical vacuum contains the one particle state with lowest energy; it is nondegenerate so long as ϕ is a scalar (spinless) field. We shall discuss the physical interpretation below.

Writing e.g., a_j for $\pi(a_j)$, we have a vacuum subspace characterized by

$$a_j|l\rangle = 0, \quad P_{jk}|l\rangle = 0,$$
$$Q_j^k|l\rangle = \delta_j^l|k\rangle + \frac{\lambda}{2}\delta_j^k|l\rangle.$$

The norm is introduced in the customary way, beginning with $\langle j|k\rangle = \delta_j^k$ (which makes π_0 hermitian), and $(x|j\rangle, x|k\rangle) = \langle j|x^* x|k\rangle$. Now

$$\langle j|a_k a^{*l}|m\rangle = \langle j|2\,Q_k^l|m\rangle = 2\,\delta_k^m \delta_j^l + \lambda_j^m \delta_k^l.$$

This is positive if $\lambda > 2$ and indefinite if $\lambda = 2$. Let

$$a^{*j}|k\rangle = |j, k\rangle.$$

In the interesting case $\lambda = 2$ the skew combination $|j, k\rangle - |k, j\rangle$ has zero norm so that the physical quotient V/V_g has only symmetric two-particle states. The three particle states are not all symmetric, however. It is easy to show that the choice $\lambda = 2$ leads to an hermitian representation on V/V_g . The proof is related to Green's ansatz. [If we adopt the point of view that the $a's$ are not observables, then we can restrict ourselves to the subspace that consists of all states with an odd number of particles. Then the hermiticity condition can be satisfied with $\lambda = 1$, in which case we fall back on the odd part of Bose-Einstein Fock space].

There is nothing inherently unphysical in the idea that the smallest number of particles is 1 rather than zero. After all, scattering usually involves at least two particles. The last, unremovable, particle may either be one of the two particles that participate in the scattering, or it may be a spectator. It would be interesting to investigate the difference (if there is any) between these two cases.

The «vacuum» sector may contain any number of particles. An interesting possibility is to associate the vacuum sector with the 3°K cosmic background radiation. This may be quite naturally connected to the existence of a singleton cosmic sea in our universe.

5. CLIFFORD QUANTIZATION

Since (para-) Bose quantization amounts to selecting a highest weight representation of $\mathrm{osp}(1,\infty)$ it is natural to ask whether other infinite superalgebras can be used. The possibility of using other (super-) algebras has been pointed out by Palev [14], but the most immediate generalizations are $\mathrm{osp}(N, 2n)$ and $\mathrm{osp}(N,\infty)$, the «extended orthosymplectic» algebras.

A basis for the odd part of $\mathrm{osp}(N,\infty)$ is $(a_j^\alpha, a^{*j\alpha})$, $j = 1, 2, \ldots;\ \alpha = 1, \ldots, N$. The even part is $\mathrm{so}(N) \oplus \mathrm{sp}(\infty)$, with basis $M^{\alpha\beta}(= -M^{\beta\alpha}), P_{jk}, P^{*jk}$ and Q_j^k . The (anti-) commutators are

$$\frac{1}{2}[\,a_j^\alpha, a^{*k\beta}]_+ = \delta_j^k M^{\alpha\beta} + \delta^{\alpha\beta} Q_j^k,$$

$$\frac{1}{2}[\,a_j^\alpha, a_k^\beta]_+ = \delta^{\alpha\beta} P_{jk},$$

$$[\,Q_j^k, a^{*l\alpha}]_- = \delta_j^l a^{*k\alpha},$$

$$[\,M^{\alpha\beta}, a_j^\gamma]_- = \delta^{\beta\gamma} a_j^\alpha - \delta^{\alpha\gamma} a_j^\beta,$$

and so on.

The most degenerate hermitian representation is the supersingleton, where

$$a_j^\alpha = \gamma^\alpha \otimes a_j, \quad a^{*j\alpha} = \gamma^\alpha \otimes a^{*j},$$

with (γ^α) , $\alpha = 1,\ldots,N$, the generators of a Clifford algebra

$$[\gamma^\alpha,\gamma^\beta]_+ = 2\,\delta^{\alpha\beta}, \quad 4\,M^{\alpha\beta} = [\gamma^\alpha,\gamma^\beta]_-$$

and $P_{jk} = \frac{1}{2}[a_j, a_k]$. The module is a direct product of a Clifford module and ordinary Fock space. The «vacuum» is degenerate (an so(N) spinor), but this does not represent a real difficulty. The quantum fields are just $\gamma^\alpha\phi(x)$; a single conventional Bose-Einstein field multiplied by N -by- N matrices, which makes this particular representation uninteresting. [Analogous extended super singleton representations of osp(N,4) appear in supergravity [10]].

However, the field operators are not so simple in other representations of osp(N,∞). Let us examine the case of a nondegenerate vacuum, characterized by

$$a_j^\alpha|0\rangle = 0, \quad Q_j^k|0\rangle = \frac{\lambda}{2}\delta_j^k|0\rangle,$$
$$P_{jk}|0\rangle = 0, \quad M^{\alpha\beta}|0\rangle = 0.$$

We have

$$\begin{aligned}
&\langle 0|a_j^\alpha a_k^\beta a^{*l\gamma}a^{*m\delta}|0\rangle = \\
&= \lambda\{\lambda(\delta^{\beta\gamma}\delta^{\alpha\delta}\delta_k^l\delta_j^m - \delta^{\beta\delta}\delta^{\alpha\gamma}\delta_k^m\delta_j^l) + \\
&+ 2(\delta^{\beta\gamma}\delta^{\alpha\delta}\delta_k^m\delta_j^l - \delta^{\beta\delta}\delta^{\alpha\gamma}\delta_k^l\delta_j^m) + \\
&+ 2\,\delta^{\alpha\beta}\delta^{\gamma\delta}\delta_k^l\delta_j^m\}.
\end{aligned}$$

If $N > 1$, then this is positive definite only if $\lambda > 2$. The most interesting case is $\lambda = 2$, in which case the subspace with zero norm includes

$$\{a^{*l\gamma}a^{*m\delta} + (\gamma,\delta) - \text{trace}\}|0\rangle$$

and

$$\{a^{*l\gamma}a^{*m\delta} - (l,m) - \text{trace}\}|0\rangle$$

The traces refer to the index pair γ,δ . Thus, in V/V_g ,

$$\left\{a^{*l\gamma}a^{*m\delta} - (l,m) - \frac{1}{N}\delta^{\gamma\delta}\sum_\alpha(a^{*l\alpha}a^{*m\alpha} - (l,m))\right\}|0\rangle = 0$$
$$\left\{a^{*l\gamma}a^{*m\delta} + (\gamma,\delta) - \frac{2}{N}\delta^{\gamma\delta}\sum_\alpha a^{*l\alpha}a^{*m\alpha}\right\}|0\rangle = 0.$$

Notice that this says nothing if $N = 1$. Since the lowest value of λ is 2 when $N > 1$, the new Fock space may be considered as a generalization of the simplest form of para-Bose Fock space. Hermiticity of this representation is proved in the Appendix.

Instead of $\mathrm{osp}(N,\infty)$ one may consider $\mathrm{osp}(3,1;\infty)$, in which $\mathrm{so}(N)$ is replaced by the Lorentz algebra. If parastatistics is natural for confined quarks, then this would seem to be an interesting possibility for confined gluons.

All of this can be repeated for fermions. It may be pointed out that the supermultiplets of supersymmetry may also be approached from the viewpoint of parastatistics, and that the validity of $[b,f]_- = 0$, b = boson, f = fermion, would then be abandoned along with $[b,b']_- = 0$ and $[f,f']_+ = 0$. Apparently, nobody has yet looked at para-super statistics. Finally, string theories may be viewed as an attempt to replace, in two-dimensional conformal field theory, $\mathrm{osp}(1,\infty)$ by a Kac-Moody algebra.

6. A REMARK ABOUT GAUGE THEORIES

So far, our discussion has made almost no use of the fact that physical theories incorporate a notion of space-time symmetry. [Without space-time symmetry, there can be no sensible physical theory]. But, in gauge theories, at least, the space-time symmetries intrude on the problem of quantization. The most direct way to see this is to consider the definition of the norm.

We have required that the norm be positive. But this is not a property that is respected by the invariant norm used for the quantization of gauge fields. In particular, in the case of Bose-Einstein quantization, the space of one-particle states is a triplet representation of the space time symmetry group (Gupta-Bleuler triplet):

$$V' \supseteq V \supseteq V_g,$$

in which V_g is the space of gauge modes, V/V_g is the space of physical states and V'/V is the space of «scalar» modes (canonically conjugate to V_g). The only reason why one cannot fix a «physical» subspace of V is that this is incompatible with the action on V of the space-time symmetry group. Therefore, without reference to this group one has no gauge theories.

The invariant metric of V' is indefinite, so the norm given by

$$\langle 0|a_j a^{*k}|0\rangle = \delta_j^k$$

is not invariant. It is therefore inevitable that the structure of V', with respect to the action of the space-time group, impinge on the discussion of quantization. Certainly, space time symmetry is more central to a physical theory than $\mathrm{osp}(1,\infty)$ symmetry.

7. COMPOSITE BOSE-EINSTEIN FIELDS

Any deviation from the simplest scheme of quantization leads to new states, as exemplified by

$$[a^{*j}, a^{*k}]_{-}|0\rangle.$$

These are introduced in addition to the symmetric states that we continue to identify and refer to as «multi-particle states». We would like to believe, or arrange the theory so, that the additional states manifest particle-like properties. [In the case when the $a^{*\prime}s$ create fermions we could speak of «bosonization» if the new states obey Bose-Einstein statistics]. To be precise, let operators b_{jk} be defined by

$$[a_j, a_k]_{-} = b_{jk}, \quad [a^{*j}, a^{*k}]_{-} = -b^{*jk}.$$

We want to know whether these operators could satisfy commutation relations of the form

$$[b_{jk}, b_{lm}] = 0, \quad [b_{jk}, b^{*lm}] = \text{number}.$$

The answer is yes, but not within the context of ordinary parastatistics.

To simplify, let us write a_{-j} for a^{*j} and let the indices run over the negative as well as the positive integers from now on. Let ω be the sympletic form of Bose-Einstein quantization:

$$[a_j, a_k]_{-} = \omega_{jk} \qquad \text{(Bose-Einstein)}. \tag{7.1}$$

As usual, we shall suppose that suppose that ω_{jk} is a complex number. A nonlinear transformation of the field variables would make ω dependent on the $a's$ and all that could then be said of ω is that it is a closed 2 -form. This reminds us of the metric tensor of special relativity – in terms of general coordinates it is not constant, though the curvature vanishes. Now general relativity and gravitons is the result of allowing the metric tensor to develop a life of its own. The new degrees appear when the metric field is no longer restricted by the requirement that it be reducible to the Minkowski metric in some special systems of coordinates. We suggest that the sympletic form ω may also be liberated.

Thus we suppose that (7.1) be replaced by («deformed» to)

$$[a_j, a_k]_{-} = \omega_{jk} + b_{jk}.$$

This introduces new states (besides those of ordinary Bose-Einstein Fock space); in particular, the states

$$b_{jk}|0\rangle, \quad b_{jk}b_{lm}|0\rangle, \quad \dots$$

We want these states to obey Bose-Einstein statistics, so we postulate that [15]

$$[b_{jk}, b_{lm}]_- = \varepsilon_{jk,lm},$$

in which ε is a numerical 2 -form. We need to known $[a_j, b_{kl}]$, and this cannot vanish because that would violate the Jacobi identity, so we put

$$[a_j, b_{kl}]_- = \Omega_{jklm} z^m,$$

with numerical coefficients Ω and operators z. The Jacobi identity requires that $[a_j, z^k] \neq 0$; the simplest possibility is

$$[a_j, z^k]_- = \delta_j^k, \quad [b_{jk}, z^l]_- = 0, \quad [z^k, z^l] = 0.$$

Now the Jacobi identity holds provided only that

$$\sum_{(jkl)} \Omega_{jklm} = 0,$$
$$\varepsilon_{mj,kl} = \Omega_{jklm} - (j, m).$$

We see that the commutation relations can be so chosen as to make the new quanta behave precisely like conventional Bose-Einstein particles. This quantization scheme is an essential part of our construction of a completely dynamical theory, composite QED in De Sitter space [18], [15]. The constituents are the famous Dirac singletons, and the additional, antisymmetric states are just photons.

The original excitations, the a -quanta, are of course unconventional, here as in parastatistics. These quanta have to be «confined»; this is the last subject on which we should like to make some remarks.

8. CONFINEMENT

When Greenberg first proposed [16] that quarks may obey parastatistics, he suggested that these «particles» should be discovered in the laboratory. However, subsequent investigations showed that locality imposes severe selection rules that tend to cast a bit of doubt on this interpretation. By the time that Gell-Mann proposed [17] replacing para by color (harking back to Green's ansatz), it was no longer expected that quarks would show up directly in experiments, and «confinement» of quarks soon became a cornerstone of strong interaction theory. It seems to us that confinement is a necessary complement to parastatistics. More generally, we would expect that all unconventional quantization schemes require confinement for the preservation of micro causality.

The usual formulation of QCD looks like a conventional quantum field theory; it is the dynamics that fundamentally alters the structure and is made responsible for the confinement of quarks and gluons. One is thus denied the hope of getting relevant information from perturbation theory. We believe that it would be better to build confinement into the very fabric of the theory; that is, into the kinematics. In fact, let us look at the relationship between parastatistics (or color) and confinement in the other direction.

By confinement, of quarks, for example, let us mean the operational fact that they cannot be isolated experimentally. This says that the quark field, if it is a useful concept at all, cannot be a local observable. Certain bilinears in the field may be observable; they will be interpreted in terms of hadrons. This fact, that the quark field is not a local observable, presents us with an opportunity. Namely, we cannot easily be convinced that quarks need to be quantized in the manner of conventional fermions. This way of looking at confinement suggests that we begin by investigating «particles» or fields that are confined already in the free state, before any interactions are contemplated; this is what we want to call «kinematic confinement». A concrete realization of this idea is offered by singleton field theory. These fields are confined for kinematical reasons; the interactions are severely restricted by gauge principles arising out the requirement of unitarity. In fact, physically interesting interactions can be introduced only if we are willing to adopt an unconventional quantization scheme. The theory reviewed in Section 7, with the operators a_j creating and destroying singletons, leads to a formulation of QED in which photons appear as states consisting of two singletons; while the singletons themselves remain unobservable [18], [14]. This construction may serve as a paradigm for the more ambitious hope of achieving something along similar lines for the strong interactions.

Basically, «confinement» of a field amounts to the lack of local observability, which in turns means that it does not interact locally. Such a field might propagate freely in some domain, but become observable on the boundary. It would be interesting to attempt to understand superconductivity in these terms.

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APPENDIX

Some of the highest weight modules discussed in the text can be proven unitarizable by a simple argument that is related to Green's ansatz. Consider first the direct product of two ordinary Bose Fock spaces. This is a unitarizable, highest weight module, with highest weight vector $|0\rangle \otimes |0\rangle$. This vector belongs to the symmetric part of the direct product. The antisymmetric part of the direct product is also a unitarizable, highest weight module, and here the vacuum sector is spanned by

$$|0\rangle \otimes |l\rangle - |l\rangle \otimes |0\rangle. \tag{A.1}$$

The destruction operators of the direct product are $a_j \otimes 1 + 1 \otimes a_j = a_j^{(1)} + a_j^{(2)}$, the two terms anticommuting with each other; the relation to Green's ansatz is evident. The highest weight module generated from the vacuum sector (A.1) is the module considered in Section 4; with $\lambda = 2$. Being an invariant submodule of a unitarizable module it is evidently unitarizable. Returning to the symmetric part of the direct product we note in the same way that $|0\rangle \otimes |0\rangle$ is cyclic for a submodule that is equivalent to the Fock space of para-Bose statistics with $\lambda = 2$; the latter is therefore unitarizable.

Exactly the same argument is used to prove unitarizability in the case of the Clifford Fock space (with $\lambda = 2$), considered in Section 5.

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